

BOUNDS FOR ELLIPTIC OPERATORS IN WEIGHTED SPACES

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Received 24 November 2004; Accepted 28 September 2005

Some estimates for solutions of the Dirichlet problem for second-order elliptic equations are obtained in this paper. Here the leading coefficients are locally VMO functions, while the hypotheses on the other coefficients and the boundary conditions involve a suitable weight function.

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1. Introduction

Let Ω be a bounded open subset of \mathbb{R}^n , $n \geq 3$, and let

$$L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a(x) \quad (1.1)$$

be a uniformly elliptic operator with measurable coefficients in Ω . Several bounds for the solutions of the problem

$$\begin{aligned} Lu &\geq f, \quad f \in L^p(\Omega), \\ u &\in W^{2,p}(\Omega) \cap C^0(\bar{\Omega}), \\ u|_{\partial\Omega} &\leq 0, \end{aligned} \quad (D)$$

($p \in]n/2, +\infty[$) have been given, and the application of such estimates allows to obtain certain uniqueness results for (D).

For instance, if $p \geq n$, $a_i, a \in L^p(\Omega)$ (with $a \leq 0$), a classical result of Pucci [4] shows that any solution u of the problem (D) verifies the bound

$$\sup_{\Omega} u \leq K \|f\|_{L^p(\Omega)}, \quad (1.2)$$

where $K \in \mathbb{R}_+$ depends on Ω , n , p , $\|a_i\|_{L^p(\Omega)}$ and on the ellipticity constant.

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The case $p < n$, where additional hypotheses on the leading coefficients are necessary, has been studied by several authors. Recently, a uniqueness result has been obtained in [3] under the assumption that the a_{ij} 's are of class VMO, $a_i = a = 0$ and $p \in]1, +\infty[$. This theorem has been extended to the case $a_i \neq 0, a \neq 0$ in [7].

If Ω is an arbitrary open subset of \mathbb{R}^n and $p \in]n/2, +\infty[$, a bound of type (1.2) and a consequent uniqueness result can be found in [1]. In fact, it has been proved there that if the coefficients a_{ij} are bounded and locally VMO, the coefficients a_i, a satisfy suitable summability conditions and $\text{ess sup}_\Omega a < 0$, then for any solution u of the problem

$$\begin{aligned} Lu &\geq f, \quad f \in L_{\text{loc}}^p(\Omega), \\ u &\in W_{\text{loc}}^{2,p}(\Omega) \cap C^0(\bar{\Omega}), \\ u|_{\partial\Omega} &\leq 0, \\ \limsup_{|x| \rightarrow +\infty} u(x) &\leq 0 \quad \text{if } \Omega \text{ is unbounded,} \end{aligned} \tag{D'}$$

there exist a ball $B \subset\subset \Omega$ and a constant $c \in \mathbb{R}_+$ such that

$$\sup_\Omega u \leq c \left(\int_B |f^-|^p dx \right)^{1/p}, \tag{1.3}$$

where f^- is the negative part of f ,

$$\int_B |f^-|^p dx = \frac{1}{|B|} \int_B |f^-|^p dx, \tag{1.4}$$

and c depends on n, p , on the ellipticity constant, and on the regularity of the coefficients of L .

The aim of this paper is to study a problem similar to that considered in [1], but with boundary conditions depending on an appropriate weight function. More precisely, fix a weight function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega)$ (see Section 2 for the definition of $\mathcal{A}(\Omega)$) and $s \in \mathbb{R}$, we consider a solution u of the problem

$$\begin{aligned} Lu &\geq f, \quad f \in L_{\text{loc}}^p(\Omega), \\ u &\in W_{\text{loc}}^{2,p}(\Omega), \\ \limsup_{x \rightarrow x_0} \sigma^s(x) u(x) &\leq 0 \quad \forall x_0 \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \sigma^s(x) u(x) &\leq 0 \quad \text{if } \Omega \text{ is unbounded.} \end{aligned} \tag{1.5}$$

If the coefficients a_{ij} are bounded and locally VMO, the functions σa_i and $\sigma^2 a$ are bounded and $\text{ess sup}_\Omega \sigma^2 a < 0$, we will prove that there exist a ball $B \subset\subset \Omega$ and a constant $c_o \in \mathbb{R}_+$ such that

$$\sup_\Omega \sigma^s u \leq c_o \left(\int_B |\sigma^{s+2} f^-|^p dx \right)^{1/p}, \tag{1.6}$$

where c_o depends on n, p, s, σ , on the ellipticity constant, and on the regularity of the coefficients of L . As a consequence, some uniqueness results are also obtained.

2. Notation and function spaces

Let Ω be an open subset of \mathbb{R}^n and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω . For each $E \in \Sigma(\Omega)$, we denote by $|E|$ the Lebesgue measure of E and put

$$E(x, r) = E \cap B(x, r) \quad \forall x \in \mathbb{R}^n, \forall r \in \mathbb{R}_+, \quad (2.1)$$

where $B(x, r)$ is the open ball in \mathbb{R}^n of radius r centered at x .

Denote by $\mathcal{A}(\Omega)$ the class of measurable functions $\rho : \Omega \rightarrow \mathbb{R}_+$ such that

$$\beta^{-1}\rho(y) \leq \rho(x) \leq \beta\rho(y) \quad \forall y \in \Omega, \forall x \in \Omega(y, \rho(y)), \quad (2.2)$$

where $\beta \in \mathbb{R}_+$ is independent of x and y . For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \left\{ z \in \partial\Omega : \lim_{x \rightarrow z} \rho(x) = 0 \right\}. \quad (2.3)$$

It is known that

$$\rho \in L_{\text{loc}}^\infty(\bar{\Omega}), \quad \rho^{-1} \in L_{\text{loc}}^\infty(\bar{\Omega} \setminus S_\rho), \quad (2.4)$$

and, if $S_\rho \neq \emptyset$,

$$\rho(x) \leq \text{dist}(x, S_\rho) \quad \forall x \in \Omega \quad (2.5)$$

(see [2, 6]). Having fixed $\rho \in \mathcal{A}(\Omega)$ such that $S_\rho = \partial\Omega$, it is possible to find a function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to ρ and such that

$$\sigma \in L_{\text{loc}}^\infty(\bar{\Omega}), \quad \sigma^{-1} \in L_{\text{loc}}^\infty(\Omega), \quad (2.6)$$

$$\sigma(x) \leq \text{dist}(x, \partial\Omega) \quad \forall x \in \Omega, \quad (2.7)$$

$$|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega, \forall \alpha \in \mathbb{N}_o^n, \quad (2.8)$$

$$\gamma^{-1}\sigma(y) \leq \sigma(x) \leq \gamma\sigma(y) \quad \forall y \in \Omega, \forall x \in \Omega(y, \sigma(y)), \quad (2.9)$$

where $c_\alpha, \gamma \in \mathbb{R}_+$ are independent of x and y (see [6]). For more properties of functions of $\mathcal{A}(\Omega)$ we refer to [2, 6].

If Ω has the property

$$|\Omega(x, r)| \geq Ar^n \quad \forall x \in \Omega, \forall r \in]0, 1], \quad (2.10)$$

where A is a positive constant independent of x and r , it is possible to consider the space $\text{BMO}(\Omega, t)$, $t \in \mathbb{R}_+$, of functions $g \in L_{\text{loc}}^1(\bar{\Omega})$ such that

$$[g]_{\text{BMO}(\Omega, t)} = \sup_{\substack{x \in \Omega \\ r \in]0, t]}} \int_{\Omega(x, r)} \left| g - \int_{\Omega(x, r)} g \right| dy < +\infty, \quad (2.11)$$

where $\int_{\Omega(x, r)} g dy = 1/|\Omega(x, r)| \int_{\Omega(x, r)} g dy$. If $g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, t_A)$, where

$$t_A = \sup \left\{ t \in \mathbb{R}_+ : \sup_{\substack{x \in \Omega \\ r \in]0, t]}} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right\}, \quad (2.12)$$

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we will say that $g \in \text{VMO}(\Omega)$ if $[g]_{\text{BMO}(\Omega,t)} \rightarrow 0$ for $t \rightarrow 0^+$. A function $\eta[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $\text{VMO}(\Omega)$ if

$$\begin{aligned} \text{BMO}(\Omega,t) &\leq \eta[g](t) \quad \forall t \in \mathbb{R}_+, \\ \lim_{t \rightarrow 0^+} \eta[g](t) &= 0. \end{aligned} \quad (2.13)$$

We say that $g \in \text{VMO}_{\text{loc}}(\Omega)$ if $(\zeta g)_o \in \text{VMO}(\mathbb{R}^n)$ for any $\zeta \in C_o^\infty(\Omega)$, where $(\zeta g)_o$ denotes the zero extension of ζg outside of Ω . A more detailed account of properties of the above defined spaces $\text{BMO}(\Omega)$ and $\text{VMO}(\Omega)$ can be found in [5].

3. An a priori bound

Fix $p \in]n/2, +\infty[$. Let B be an open ball of \mathbb{R}^n , $n \geq 3$, of radius δ . We consider in B the differential operator

$$L_B = \sum_{i,j=1}^n \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i} + \alpha(x), \quad (3.1)$$

with the following condition on the coefficients:

$$\begin{aligned} \alpha_{ij} &= \alpha_{ji} \in L^\infty(B) \cap \text{VMO}(B), \quad i, j = 1, \dots, n, \\ \exists \mu \in \mathbb{R}_+ : \sum_{i,j=1}^n \alpha_{ij} \zeta_i \zeta_j &\geq \mu |\zeta|^2 \quad \text{a.e. in } B, \quad \forall \zeta \in \mathbb{R}^n, \\ \alpha_i &\in L^\infty(B), \quad i = 1, \dots, n, \quad \alpha \in L^\infty(B), \quad \alpha \leq 0 \text{ a.e. in } B. \end{aligned} \quad (h_B)$$

Let $\mu_0, \mu_1, \mu_2 \in \mathbb{R}_+$ such that

$$\sum_{i,j=1}^n \|\alpha_{ij}\|_{L^\infty(B)} \leq \mu_0, \quad \delta \sum_{i=1}^n \|\alpha_i\|_{L^\infty(B)} \leq \mu_1, \quad \delta^2 \|\alpha\|_{L^\infty(B)} \leq \mu_2. \quad (3.2)$$

Note that under the assumption (h_B) , the operator L_B from $W^{2,p}(B)$ into $L^p(B)$ is bounded and the estimate

$$\|L_B u\|_{L^p(B)} \leq c_1 \|u\|_{W^{2,p}(B)} \quad \forall u \in W^{2,p}(B) \quad (3.3)$$

holds, where $c_1 \in \mathbb{R}_+$ depends on $n, p, \mu_0, \mu_1, \mu_2$.

LEMMA 3.1. *Suppose that condition (h_B) is verified, and let u be a solution of the problem*

$$\begin{aligned} u &\in W^{2,p}(B), \\ L_B u &\geq \phi, \quad \phi \in L^p(B), \\ u|_{\partial B} &\leq 0. \end{aligned} \quad (3.4)$$

Then there exists $c \in \mathbb{R}_+$ such that

$$\sup_B u \leq c \delta^{2-n/p} \|\phi^-\|_{L^p(B)}, \quad (3.5)$$

where c depends on $n, p, \mu, \mu_0, \mu_1, \mu_2, [p(\alpha_{ij})]_{\text{BMO}(\mathbb{R}^n, \cdot)}$, and where $p(\alpha_{ij})$ is an extension of α_{ij} to \mathbb{R}^n in $L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n)$.

Proof. Put $B = B(y, \delta)$, where y is the centre of B , and $B^* = B(y, 1)$.

Consider the function $T : B \rightarrow B^*$ defined by the position

$$T(x) = y + \frac{x - y}{\delta} = z, \quad (3.6)$$

and for each function g defined on B , put $g^* = g \circ T^{-1}$.

We observe that

$$L_B^* u^* = \delta^2 (L_B u)^*, \quad (3.7)$$

where

$$L_B^* = \sum_{i,j=1}^n \alpha_{ij}^*(z) \frac{\partial^2}{\partial z_i \partial z_j} + \delta \sum_{i=1}^n \alpha_i^*(z) \frac{\partial}{\partial z_i} + \delta^2 \alpha^*(z). \quad (3.8)$$

Denote by $p(\alpha_{ij})$ an extension of α_{ij} to \mathbb{R}^n such that

$$p(\alpha_{ij}) \in L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n) \quad (3.9)$$

(for the existence of such function see [5, Theorem 5.1]). Since

$$p(\alpha_{ij})^* \in L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n), \quad p(\alpha_{ij})_{|B^*}^* = \alpha_{ij}^*, \quad (3.10)$$

it follows that

$$\alpha_{ij}^* \in L^\infty(B^*) \cap \text{VMO}(B^*). \quad (3.11)$$

Moreover, the condition (h_B) yields that

$$\begin{aligned} \alpha_{ij}^* &= \alpha_{ji}^*, \quad i, j = 1, \dots, n, \\ \sum_{i,j=1}^n \alpha_{ij}^* \zeta_i \zeta_j &\geq \mu |\zeta|^2 \quad \text{a.e. in } B^*, \quad \forall \zeta \in \mathbb{R}^n, \\ \alpha_i^* &\in L^\infty(B^*), \quad i = 1, \dots, n, \quad \alpha^* \in L^\infty(B^*), \quad \alpha^* \leq 0 \quad \text{a.e. in } B^*. \end{aligned} \quad (3.12)$$

We observe that the condition (3.12) implies that for $r, s \in]1, +\infty[$ the modulus of continuity of $\delta \alpha_i^*$ in $L^r(B^*)$ and that of $\delta^2 \alpha^*$ in $L^s(B^*)$ depend only on $\|\delta \alpha_i^*\|_{L^\infty(B^*)}$ and $\|\delta^2 \alpha^*\|_{L^\infty(B^*)}$, respectively.

Thus, applying (3.10), (3.12), and [7, Theorem 2.1], it follows that the problem

$$\begin{aligned} L_B^* v &= \psi \in L^p(B^*), \\ v &\in W^{2,p}(B^*) \cap \overset{\circ}{W}^{1,p}(B^*) \end{aligned} \quad (3.13)$$

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has a unique solution v satisfying the estimate

$$\|v\|_{W^{2,p}(B^*)} \leq K \|\psi\|_{L^p(B^*)}, \quad (3.14)$$

where K depends on $n, p, \mu, \mu_0, \mu_1, \mu_2, [p(\alpha_{ij})^*]_{\text{BMO}(R^n, \cdot)}$.

The estimate (3.5) follows now from (3.14) using the same arguments of the proof of Lemma 3.2 [1] in order to obtain there (e_B) from [1, (3.23)]. \square

4. Hypotheses and preliminary results

Let Ω be an open subset of \mathbb{R}^n , $n \geq 3$. Fix $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ such that $S_\rho = \partial\Omega$.

Consider a function $g \in C_0^\infty(\mathbb{R}_+)$ satisfying the condition

$$0 \leq g \leq 1, \quad g(t) = 1 \quad \text{if } t \geq 1, \quad g(t) = 0 \quad \text{if } t \leq \frac{1}{2}. \quad (4.1)$$

For any $k \in \mathbb{N}$, we put

$$\eta_k(x) = \frac{1}{k} \zeta_k(x) + (1 - \zeta_k(x)) \sigma(x), \quad x \in \Omega, \quad (4.2)$$

where $\zeta_k(x) = g(k\sigma(x))$, $x \in \Omega$. Clearly, $\eta_k \in C^\infty(\Omega)$ for any $k \in \mathbb{N}$ and

$$\eta_k(x) = \begin{cases} \frac{1}{k} & \text{if } x \in \bar{\Omega}_k, \\ \sigma(x) & \text{if } x \in \Omega \setminus \Omega_{2k}, \end{cases} \quad (4.3)$$

where

$$\Omega_k = \left\{ x \in \Omega : \sigma(x) > \frac{1}{k} \right\}, \quad k \in \mathbb{N}. \quad (4.4)$$

In the following we will use the notation

$$f_x = \left(\sum_{i=1}^n f_{x_i}^2 \right)^{1/2}, \quad f_{xx} = \left(\sum_{i,j=1}^n f_{x_i x_j}^2 \right)^{1/2}. \quad (4.5)$$

It is easy to show that for each $k \in \mathbb{N}$,

$$\sigma(x) \leq \eta_k(x) \leq 2\sigma(x), \quad x \in \Omega \setminus \bar{\Omega}_k, \quad (4.6)$$

$$c'_k \sigma(x) \leq \eta_k(x) \leq \sigma(x), \quad x \in \Omega_k, \quad (4.7)$$

$$(\eta_k(x))_x \leq c_1 (\sigma(x))_x, \quad x \in \Omega, \quad (4.8)$$

$$(\eta_k(x))_{xx} \leq c_2 \frac{(\sigma(x))_x^2 + \sigma(x)(\sigma(x))_{xx}}{\sigma(x)}, \quad x \in \Omega, \quad (4.9)$$

where $c'_k \in \mathbb{R}_+$ depends on k and σ , and $c_1, c_2 \in \mathbb{R}_+$ depend only on n . Moreover, for any $s \in \mathbb{R}$, we have

$$\frac{(\eta_k^s(x))_x}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x}{\sigma(x)}, \quad x \in \Omega, \quad (4.10)$$

$$\frac{(\eta_k^s(x))_{xx}}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x^2 + \eta_k(x)(\eta_k(x))_{xx}}{\sigma^2(x)}, \quad x \in \Omega, \quad (4.11)$$

where $c_3 \in \mathbb{R}_+$ depends on s and n .

We consider in Ω the differential operator

$$L = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a(x), \quad (4.12)$$

and put

$$L_o = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \quad (4.13)$$

We will make the following assumption on the coefficients of L :

$$\begin{aligned} a_{ij} &= a_{ji} \in L^\infty(\Omega) \cap \text{VMO}_{\text{loc}}(\Omega), \quad i, j = 1, \dots, n, \\ \exists \nu, \nu_0 \in \mathbb{R}_+ : \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} &\leq \nu_0, \quad \sum_{i,j=1}^n a_{ij} \zeta_i \zeta_j \geq \nu |\zeta|^2 \quad \text{a.e. in } \Omega, \quad \forall \zeta \in \mathbb{R}^n, \\ \exists \nu_1, \nu_2 \in \mathbb{R}_+ : \text{ess sup}_\Omega \left(\sigma(x) \sum_{i=1}^n |a_i(x)| \right) &\leq \nu_1, \quad \text{ess sup}_\Omega (\sigma^2(x) |a(x)|) \leq \nu_2, \\ \exists a_o \in \mathbb{R}_+ : \text{ess sup}_\Omega (\sigma^2(x) a(x)) &= -a_o. \end{aligned} \quad (h_1)$$

Fixed $s \in \mathbb{R}$, let u be a solution of the problem

$$\begin{aligned} Lu &\geq f, \quad f \in L_{\text{loc}}^p(\Omega), \quad u \in W_{\text{loc}}^{2,p}(\Omega), \\ \limsup_{x \rightarrow x_o} \sigma^s(x) u(x) &\leq 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \sigma^s(x) u(x) &\leq 0 \quad \text{if } \Omega \text{ is unbounded.} \end{aligned} \quad (P)$$

For any $k \in \mathbb{N}$, we put

$$w_k(x) = \eta_k^s(x) u(x), \quad x \in \Omega. \quad (4.14)$$

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LEMMA 4.1. *Suppose that condition (h_1) holds. Then, for any $k \in \mathbb{N}$ there exist functions b_i^k ($i = 1, \dots, n$), b^k , g^k and positive constants β_1 and β_2 such that*

$$\operatorname{ess\,sup}_{\Omega} \left(\sigma(x) \sum_{i=1}^n |b_i^k(x)| \right) \leq \beta_1, \quad (4.15)$$

$$\operatorname{ess\,sup}_{\Omega} (\sigma^2(x) |b^k(x)|) \leq \beta_2, \quad (4.16)$$

$$g^k \in L_{\operatorname{loc}}^p(\Omega), \quad (4.17)$$

where β_1 depends on s , n , ν_0 , ν_1 and β_2 depends on s , n , ν_0 , ν_2 . Moreover, the function w_k , $k \in \mathbb{N}$, satisfies the following conditions:

$$\begin{aligned} w_k &\in W_{\operatorname{loc}}^{2,p}(\Omega), \quad \limsup_{x \rightarrow x_0} w_k(x) \leq 0 \quad \forall x_0 \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} w_k(x) &\leq 0 \quad \text{if } \Omega \text{ is unbounded,} \end{aligned} \quad (4.18)$$

$$L_o w_k + \sum_{i=1}^n b_i^k (w_k)_{x_i} + b^k w_k \geq g^k \quad \text{in } \Omega. \quad (4.19)$$

Proof. Fix $k \in \mathbb{N}$. From (4.6)–(4.11) and from (2.6), (2.8), it easily follows that the function w_k , defined by (4.14), verifies (4.18).

Moreover, observe that

$$\begin{aligned} L_o w_k - u L_o \eta_k^s - 2 \sum_{i,j=1}^n a_{ij} (\eta_k^s)_{x_j} u_{x_i} + \sum_{i=1}^n a_i (\eta_k^s u)_{x_i} \\ - u \sum_{i=1}^n a_i (\eta_k^s)_{x_i} + a \eta_k^s u = \eta_k^s L u, \quad x \in \Omega. \end{aligned} \quad (4.20)$$

Since

$$(\eta_k^s)_{x_j} u_{x_i} = (\eta_k^s u)_{x_i} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} - \frac{(\eta_k^s)_{x_i} (\eta_k^s)_{x_j}}{(\eta_k^s)^2} (\eta_k^s u), \quad (4.21)$$

from (4.20), (4.19) follows, where we have put

$$\begin{aligned} b_i^k &= a_i - 2 \sum_{j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s}, \quad i = 1, \dots, n, \\ b^k &= a + 2 \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i} (\eta_k^s)_{x_j}}{(\eta_k^s)^2} - \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i x_j}}{\eta_k^s}, \\ g^k &= \eta_k^s f + \sum_{i=1}^n a_i \frac{(\eta_k^s)_{x_i}}{\eta_k^s} w_k. \end{aligned} \quad (4.22)$$

On the other hand, using the hypothesis (h_1) , (4.6)–(4.11), and (2.8) it is easy to show that there exist $\beta_1 \in \mathbb{R}_+$ depending on s , n , ν_0 , ν_1 and $\beta_2 \in \mathbb{R}_+$ depending on s , n , ν_0 , ν_2 , such that (4.15), (4.16), (4.17) hold. \square

Now we suppose that the following hypothesis on ρ holds:

$$\lim_{k \rightarrow +\infty} \left(\sup_{\Omega \setminus \Omega_k} ((\sigma(x))_x + \sigma(x)(\sigma(x))_{xx}) \right) = 0. \quad (h_2)$$

An example of function ρ such that σ satisfies (h_2) is provided in [2].

LEMMA 4.2. *Suppose that conditions (h_1) and (h_2) hold. Then there exists $k_o \in \mathbb{N}$ such that*

$$\begin{aligned} \operatorname{ess\,sup}_{\Omega} \left(\sigma(x) \sum_{i=1}^n |b_i^{k_o}(x)| \right) &\leq \nu_1 + \frac{a_o}{2}, \\ \operatorname{ess\,sup}_{\Omega} (\sigma^2(x) b^{k_o}(x)) &\leq -\frac{a_o}{2}, \\ g^{k_o}(x) &\geq \eta_{k_o}^s(x) f(x) - \frac{a_o}{8} \sigma^{-2}(x) |w_{k_o}(x)|, \quad x \in \Omega. \end{aligned} \quad (4.23)$$

Proof. From (4.10), (4.11), and hypothesis (h_1) , we deduce that

$$\begin{aligned} \sigma \left| \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} \right| &\leq c_4 (\eta_k)_x, \\ \sigma^2 \left| \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i} (\eta_k^s)_{x_j}}{(\eta_k^s)^2} \right| + \sigma^2 \left| \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i x_j}}{\eta_k^s} \right| &\leq c_5 ((\eta_k)_x^2 + \eta_k (\eta_k)_{xx}), \\ \sigma^2 \left| \sum_{i=1}^n a_i \frac{(\eta_k^s)_{x_i}}{\eta_k^s} \right| &\leq c_6 (\eta_k)_x, \end{aligned} \quad (4.24)$$

where $c_4, c_5 \in \mathbb{R}_+$ depend on s, n, ν_0 and $c_6 \in \mathbb{R}_+$ depends on s, n, ν_1 . Observing that $(\eta_k)_x = (\eta_k)_{xx} = 0$ in $\bar{\Omega}_k$, the statement follows now from (4.8), (4.9), (h_1) , (h_2) , and (4.24). \square

5. Main results

It is well known that there exists a function $\tilde{\alpha} \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to $\operatorname{dist}(\cdot, \partial\Omega)$ (see, e.g., [8]). For every positive integer m , we define the function

$$\psi_m : x \in \bar{\Omega} \longrightarrow g(m\tilde{\alpha}(x)) \left(1 - g\left(\frac{|x|}{2m}\right) \right), \quad (5.1)$$

where $g \in C^\infty(\bar{\mathbb{R}}_+)$ verifies (4.1). It is easy to show that ψ_m belongs to $C_o^\infty(\Omega)$ for every $m \in \mathbb{N}$ and

$$0 \leq \psi_m \leq 1, \quad \operatorname{supp} \psi_m \subseteq E_{2m}, \quad \psi_m|_{E_m} = 1, \quad (5.2)$$

where

$$E_m = \left\{ x \in \Omega : |x| < m, \tilde{\alpha}(x) > \frac{1}{m} \right\}. \quad (5.3)$$

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Remark 5.1. It follows from hypothesis (h_1) and from [5, Lemma 4.2] that for any $m \in \mathbb{N}$ the functions $(\psi_m a_{ij})_o$ (obtained as extensions of $\psi_m a_{ij}$ to \mathbb{R}^n with zero values out of Ω) belong to $\text{VMO}(\mathbb{R}^n)$ and

$$[(\psi_m a_{ij})_o]_{\text{BMO}(\mathbb{R}^n, t)} \leq [\psi_m a_{ij}]_{\text{BMO}(\Omega, t)}, \quad (5.4)$$

for t small enough.

In the following we denote by w , b_i , b , and g the functions defined by (4.14), (4.22), respectively, corresponding to $k = k_o$, where k_o is the positive integer of Lemma 4.2

We can now prove the main result of the paper.

THEOREM 5.2. *Suppose that conditions (h_1) and (h_2) hold, and let u be a solution of the problem (P). Then there exist an open ball $B \subset \subset \Omega$ and a constant $c_o \in \mathbb{R}_+$ such that*

$$\sup_{\Omega} \sigma^s(x) u(x) \leq c_o \left(\int_B |\sigma^{s+2} f^-|^p dx \right)^{1/p}, \quad (5.5)$$

where c_o depends only on $n, p, s, \gamma, \nu, \nu_0, \nu_1, \nu_2, a_o, \eta[\psi_m a_{ij}]$ ($m \in \mathbb{N}$).

Proof. It can be assumed that $\sup_{\Omega} \sigma^s(x) u(x) > 0$. Thus it follows from (4.14) and (4.18) that there exists $y \in \Omega$ such that $\sup_{\Omega} w(x) = w(y)$; moreover, there exists $R_o \in]0, \text{dist}(y, \partial\Omega)[$ such that $w(x) > 0$ for all $x \in B(y, R_o)$.

Let $\lambda, \alpha, \alpha_o \in \mathbb{R}_+$, with $\alpha_o > 1$ (that will be chosen late), such that

$$\lambda \alpha \leq \min\{R_o, \sigma(y)\}, \quad \alpha = \alpha_o \sigma(y). \quad (5.6)$$

In the following we denote by B the open ball $B(y, \alpha\lambda)$.

We put

$$\varphi(x) = 1 + \lambda^2 - \frac{|x - y|^2}{\alpha^2}, \quad x \in \bar{B}, \quad (5.7)$$

and observe that

$$1 \leq \varphi(x) \leq 1 + \lambda^2 \leq 2, \quad x \in \bar{B}, \quad (5.8)$$

$$\varphi_{x_i} \leq \frac{2\lambda}{\alpha}, \quad \varphi_{x_i} \varphi_{x_j} \leq \frac{4\lambda^2}{\alpha^2}, \quad i, j = 1, \dots, n, \quad (5.9)$$

$$\varphi_{x_i x_j} = 0 \quad \text{if } i \neq j, \quad \varphi_{x_i x_j} = -\frac{2}{\alpha^2} \quad \text{if } i = j. \quad (5.10)$$

Consider now the function v defined by

$$v(x) = \varphi(x)w(x) - w(y), \quad x \in \bar{B}. \quad (5.11)$$

Obviously,

$$v|_{\partial\Omega} = w|_{\partial\Omega} - w(y) \leq 0, \quad v(y) = \lambda^2 w(y). \quad (5.12)$$

It is easy to show that

$$\begin{aligned} L_o(\varphi w) - wL_o\varphi - 2 \sum_{i,j=1}^n a_{ij}\varphi_{x_j}w_{x_i} + \sum_{i=1}^n b_i(\varphi w)_{x_i} \\ - \sum_{i=1}^n b_i\varphi_{x_i}w + b\varphi w = \varphi \left(L_o w + \sum_{i=1}^n b_i w_{x_i} + bw \right) \geq \varphi g \quad \text{in } B. \end{aligned} \quad (5.13)$$

Thus

$$L_o(\varphi w) + \sum_{i=1}^n d_i(\varphi w)_{x_i} + d\varphi w \geq \varphi g + \sum_{i=1}^n b_i\varphi_{x_i}w \quad \text{in } B, \quad (5.14)$$

where

$$d_i = b_i - 2 \sum_{j=1}^n a_{ij} \frac{\varphi_{x_j}}{\varphi}, \quad i = 1, \dots, n, \quad (5.15)$$

$$d = b + 2 \sum_{i,j=1}^n a_{ij} \frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2} - \sum_{i,j=1}^n a_{ij} \frac{\varphi_{x_i x_j}}{\varphi}. \quad (5.16)$$

Therefore we obtain from (5.14) that

$$L_o v + \sum_{i=1}^n d_i v_{x_i} + dv \geq h, \quad (5.17)$$

where

$$h = \varphi g + w \sum_{i=1}^n b_i \varphi_{x_i} - dw(y). \quad (5.18)$$

Clearly, (2.9), (5.6), and (5.9) yield that

$$|\varphi_{x_i}| \leq 2\gamma \frac{\sigma}{\alpha_o^2 \sigma^2(y)} \quad \text{in } B, \quad (5.19)$$

and hence it follows from Lemma 4.2 that

$$\begin{aligned} h &\geq \varphi \eta_{k_o}^\varepsilon f - \frac{a_o}{8} \sigma^{-2} \varphi w(y) - 2\gamma w(y) \left(\nu_1 + \frac{a_o}{2} \right) \frac{1}{\alpha_o^2} \sigma^{-2}(y) - dw(y) \\ &\geq \varphi \eta_{k_o}^\varepsilon f + \left[-d - \left(\frac{a_o}{4} \gamma^2 + 2 \frac{\gamma \nu_1}{\alpha_o^2} + \frac{\gamma a_o}{\alpha_o^2} \right) \sigma^{-2}(y) \right] w(y). \end{aligned} \quad (5.20)$$

The constant α_o can be chosen in such a way that $d < -d_o \sigma^{-2}(y)$ in B , where

$$d_o = \frac{a_o}{4} \gamma^2 + 2 \frac{\gamma \nu_1}{\alpha_o^2} + \frac{\gamma a_o}{\alpha_o^2}. \quad (5.21)$$

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In fact, by Lemma 4.2, (5.9) and (5.10), we have

$$\begin{aligned}
 d + d_o \sigma^{-2}(y) &= b + 2 \sum_{i,j=1}^n a_{ij} \frac{\varphi_{x_i} \varphi_{x_j}}{\varphi^2} - \sum_{i,j=1}^n a_{ij} \frac{\varphi_{x_i x_j}}{\varphi} + d_o \sigma^{-2}(y) \\
 &\leq -\frac{a_o}{2} \sigma^{-2} + 8\nu_o \frac{\lambda^2}{\alpha^2} + 2\nu_o \frac{1}{\alpha^2} + d_o \sigma^{-2}(y) \\
 &\leq \left[-\gamma^2 \frac{a_o}{4} + (10\nu_o + 2\gamma\nu_1 + \gamma a_o) \frac{1}{\alpha_o^2} \right] \sigma^{-2}(y),
 \end{aligned} \tag{5.22}$$

and hence, fixed α_o such that

$$\frac{1}{\alpha_o^2} \leq \frac{\gamma^2 a_o}{4(10\nu_o + 2\gamma\nu_1 + \gamma a_o)}, \tag{5.23}$$

it follows that

$$d < -d_o \sigma^{-2}(y) \quad \text{in } B. \tag{5.24}$$

By (5.11), (5.12), and (5.15)–(5.17), we deduce that the problem

$$\begin{aligned}
 v &\in W^{2,p}(B), \\
 L_o v + \sum_{i=1}^n d_i v_{x_i} + dv &\geq \varphi \eta_{k_o}^s f, \quad f \in L^p(B), \\
 v|_{\partial B} &\leq 0
 \end{aligned} \tag{5.25}$$

satisfies the hypotheses of Lemma 3.1. Therefore, it follows from (5.6), (4.15), and (4.16) that there exists a constant $c_1 \in \mathbb{R}_+$, depending on $n, p, s, \gamma, \nu, \nu_o, \nu_1, \nu_2, [p(a_{ij}|_B)]_{\text{BMO}(R^n, \cdot)}$, such that

$$v(x) \leq c_1 (\lambda \alpha)^{2-n/p} \left\| (\varphi \eta_{k_o}^s f)^- \right\|_{L^p(B)} \quad \forall x \in B. \tag{5.26}$$

So it follows from (5.8) and from (5.26) with $x = y$ that

$$\lambda^2 w(y) \leq c_1 (\lambda \alpha)^{2-n/p} \left\| (\varphi \eta_{k_o}^s f)^- \right\|_{L^p(B)} \leq 2c_1 (\lambda \alpha)^{2-n/p} \left\| \eta_{k_o}^s f^- \right\|_{L^p(B)}. \tag{5.27}$$

Thus by (5.6) and (5.27) we have

$$w(y) \leq c_2 (\lambda \alpha)^{-n/p} \alpha_o^2 \sigma^2(y) \left\| \eta_{k_o}^s f^- \right\|_{L^p(B)} \leq c_3 (\lambda \alpha)^{-n/p} \alpha_o^2 \left\| \sigma^2 \eta_{k_o}^s f^- \right\|_{L^p(B)}, \tag{5.28}$$

where $c_2, c_3 \in \mathbb{R}_+$ depend on the same parameters as c_1 . Finally from (4.6), (4.7), (4.14), and (5.28) we obtain

$$\sup_{\Omega} \sigma^s u \leq c_4 (\lambda \alpha)^{-n/p} \left(\int_B |\sigma^{2+s} f^-|^p dx \right)^{1/p} \leq c_5 \left(\int_B |\sigma^{s+2} f^-| dx \right)^{1/p}, \tag{5.29}$$

where $c_4, c_5 \in \mathbb{R}_+$ depend on the same parameters as c_1 and on a_o . Then, if we choose

$$p(a_{ij}|_B) = (\psi_{m_1} a_{ij})_o, \quad (5.30)$$

where m_1 is a positive integer such that $\psi_{m_1}|_B = 1$, (5.5) follows from (5.29), (5.30), and from Remark 5.1. \square

COROLLARY 5.3. *Suppose that conditions (h_1) and (h_2) hold, and let u be a solution of the problem*

$$\begin{aligned} Lu &= f, \quad \sigma^{s+2} f \in L^\infty(\Omega), \quad u \in W_{\text{loc}}^{2,p}(\Omega), \\ \limsup_{x \rightarrow x_o} \sigma^s(x) u(x) &= 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \sigma^s(x) u(x) &= 0 \quad \text{if } \Omega \text{ is unbounded.} \end{aligned} \quad (p')$$

Then

$$\sup_{\Omega} \sigma^s |u| \leq c_o \|\sigma^{s+2} f\|_{L^\infty(\Omega)}, \quad (5.31)$$

where $c_o \in \mathbb{R}_+$ is the constant of the statement of Theorem 5.2.

Proof. The result can be obtained applying Theorem 5.2 to the functions u and $-u$. \square

The following uniqueness result is an obvious consequence of Corollary 5.3.

COROLLARY 5.4. *If the hypotheses (h_1) and (h_2) hold, then the problem*

$$\begin{aligned} Lu &= 0, \quad u \in W_{\text{loc}}^{2,p}(\Omega), \\ \limsup_{x \rightarrow x_o} \sigma^s(x) u(x) &= 0 \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \sigma^s(x) u(x) &= 0 \quad \text{if } \Omega \text{ is unbounded} \end{aligned} \quad (p'')$$

has only the zero solution.

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